KÄHLER DIFFERENTIALS

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1. Motivation

Recall that for a smooth manifold M we have the smooth vector bundle $\pi\colon T^*M \to M$, known as the *cotangent bundle*, whose fibers are T_p^*M; the cotangent space of M at p. To completely describe T^*M, we would also have to mention the transition functions. For a smooth k-variety X (k a field), we would like to define a similar notion. We will see that the right analog of this for schemes is the qausicoherent sheaf of *relative differentials* or *differential forms* $\Omega_{X/k}$ on X. Why is $\Omega_{X/k}$ the right notion for the cotangent bundle? In the setting of a smooth manifold M, the fiber of the vector bundle $\pi\colon T^*M \to M$ over each $p \in M$ is the cotangent space T_p^*M. Thus we should expect that the fiber of $\Omega_{X/k}$ over each k*-rational* point to be isomorphic to the cotangent space $T_x^*X = \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$. For an open subset $U \subset M$, we have the set of 1-forms $\Omega^1(U)$ which are smooth sections $\omega\colon U \to T^*M$. In particular, for a smooth function $f \in C^\infty(U)$ we have a 1-form $df\colon M \to T^*M$ given by $(df)_p(\nu) = \nu(f)$ for all $p \in M$, $\nu \in T_pM$. This is the *differential of* f. Derivations will be the right analog of this.

2. In the affine setting

Definition 1.1. *Let $\phi\colon B \to A$ be a B-algebra and M an A-module. A $B-$**derivation** of A into M is a function $\mathrm{d}\colon A \to M$ such that for all $a, a' \in A$,*

- *(additive)* $\mathrm{d}a + \mathrm{d}a' = \mathrm{d}(a + a')$
- *(Leibniz)* $\mathrm{d}(aa') = a\mathrm{d}a' + a'\mathrm{d}a$
- *(trivial on pullback)* $\mathrm{d}b = 0$*, for all* $b \in \phi(B)$.

We remark that d is B-linear, it satisfies the quotient rule: $\mathrm{d}(f/g) = (g\mathrm{d}f - f\mathrm{d}g)/g^2$, as well as the chain rule : for $f \in B[x]$ and $g \in A$, $\mathrm{d}f(g) = f'(g)\mathrm{d}g$, where f' is defined formally as : if $f = \sum b_i x^i$ then $f' = \sum b_i i x^{i-1}$

Definition 1.2. *Let A be a B-algebra. The **module of relative differentials** of A over B is the A-module $\Omega_{A/B}$ together with a B-derivation $\mathrm{d}\colon A \to \Omega_{A/B}$ which satisfies the following universal property: for any $A-$module M and $B-$derivation $\mathrm{d}'\colon A \to M$, there is a unique A-module homomorphism $\rho\colon \Omega_{A/B} \to M$ such that $\mathrm{d}' = \rho \circ \mathrm{d}$. In other words, $\Omega_{A/B}$ is initial for the structure of A-modules together with B-derivations $d\colon A \to M$.*

For existence, one can take $\Omega_{A/B}$ to be the free A-module generated by symbols $\mathrm{d}a$, $a \in A$, modulo the relations:

- $\mathrm{d}a + \mathrm{d}a' = \mathrm{d}(a + a')$
- $\mathrm{d}(aa') = a\mathrm{d}a' + a'\mathrm{d}a$
- $\mathrm{d}b = 0$, $b \in \phi(B)$.

with derivation $d\colon A \to \Omega_{A/B}$ *defined by* $a \mapsto \mathrm{d}a$.

Proposition 1.3. *If* $A = B[x_1, \ldots, x_n]$, *then* $\Omega_{A/B} \cong A^{\oplus n}$, *as modules.*

Proof. For any $f \coloneqq g(x_1, \ldots, x_n) \in A$, by combining the rules of a derivation as well as the chain rule, we can write

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i.$$

This shows that $\{\mathrm{d}x_i\}_{i=1}^n$ span $\Omega_{A/B}$ over A. We can define a module of relative differentials (Ω, d') to be the free A-module Ω generated by the symbols $\{\mathrm{d}T_i\}_{i=1}^n$ with derivation $d'\colon A \to \Omega$ given by $d'(g) = \sum_{i=1}^n \frac{\partial g}{\partial x_i} \mathrm{d}T_i$. There is a unique A-module homomorphism $\rho\colon \Omega \to \Omega_{A/B}$ given by $\mathrm{d}T_i \mapsto \mathrm{d}x_i$. So by the universal property, $\Omega \cong \Omega_{A/B}$ as modules. In particular, $\Omega_{A/B} \cong A^{\oplus n}$. □

Theorem 1.4. *Let* k *be a field,* A *a finite type* k*-algebra and* $\mathfrak{m} \subset A$ *a maximal ideal with* $A/\mathfrak{m} \cong k$. *Then* $\mathfrak{m}/\mathfrak{m}^2$ *is isomorphic to* $\Omega_{A/k} \otimes_A k$ *as* k*-vector spaces; here* k *inherits an* A*-module structure via* $k \cong A/\mathfrak{m}$.

Proof. In this case, $A \cong B/I$, where $B \coloneqq k[x_1, \ldots, x_n]$. Since $k[x_1, \ldots, x_n]$ is Noetherian, I is finitely generated, say $I = (f_1, \ldots, f_r)$. From a standard result in classical algebraic geometry, we have that $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n - \operatorname{rank} J$, where $J = \left[\frac{\partial f_i}{\partial x_j}\right]$ On the other hand, by [[1], 21.2.E] we have that

$$A^{\oplus r} \xrightarrow{J} A^{\oplus n} \to \Omega_{A/k} \to 0;$$

Tensoring by k gives;

$$A^{\oplus r} \otimes_A k \xrightarrow{\widetilde{J}} A^{\oplus n} \otimes_A k \to \Omega_{A/k} \otimes_A k \to 0;$$
$$(A \otimes_A k)^{\oplus r} \xrightarrow{\widetilde{J}} (A \otimes_A k)^{\oplus n} \to \Omega_{A/k} \otimes_A k \to 0;$$

i.e $\Omega_{A/k} \otimes_A k$ is the cokernel of the induced map $\widetilde{J}$. Since

$$\dim_k \operatorname{coker}(\widetilde{J}) = n - \operatorname{rank} J,$$

we conclude that $\dim_k \Omega_{A/k} \otimes_A k = n - \operatorname{rank} J$. Thus $\mathfrak{m}/\mathfrak{m}^2$ and $\Omega_{A/k} \otimes_A k$ have the same dimension as vector spaces over k, so they must be isomorphic. □

3. Setting of a k-variety

The strategy is nothing new; we have already built all the required machinery affine locally so we naively glue these together and thus define the *sheaf* of relative differentials (or differential forms).

Theorem 1.5. *Let X be a k-variety. Then there exists a unique qausicoherent sheaf $\Omega_{X/k}$ on X such that for any affine open subset $U \subset X$, and any $x \in U$,*

$$\Omega_{X/k}\big|_U \cong \widetilde{\Omega_{\mathcal{O}_X(U)/k}} \quad \textit{and} \quad \big(\Omega_{X/k}\big)_x \cong \Omega_{\mathcal{O}_{X,x}/k}.$$

Therefore, for a k-variety X we call $\Omega_{X/k}$ the ***sheaf of relative differentials (or differential forms)*** *of X.*

For a proof, see [[2] Proposition 6.1.17]. Notice that the hypothesis that X is a k-variety is not needed for existence. Does everything check out as expected for a k-variety X? Say $X = \operatorname{Spec} A$, where $A = k[x_1, \dots, x_n]/(f_1, , \dots, f_r)$. Then

- For each k-rational point $x \in X$, we have that the fiber of $\Omega_{X/k}$ over x is
$$(\Omega_{X/k})_x \otimes_{\mathcal{O}_{X,x}} k(x) \cong \Omega_{\mathcal{O}_{X,x}/k} \otimes_{\mathcal{O}_{X,x}} k,$$
where $k(x) \coloneqq \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} \cong k$. Since X is a k-variety, $\mathcal{O}_{X,x}$ is a finite type k-algebra, and thus applying our theorem from before we have that $\Omega_{\mathcal{O}_{X,x}/k} \otimes_{\mathcal{O}_{X,x}} k \cong \mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2$ as k-vector spaces.
- We have that $\Omega_{X/k} \cong \widetilde{\Omega_{A/k}}$ which is the module of relative differential forms (or 1-forms), as defined earlier. In particular, for any $f \in A$, we have its corresponding differential $\mathrm{d}f$.

References

[1] Ravi Vakil (2023) *Foundations of Algebraic Geometry*, Version 1.0, Stanford University. Found at: https://math.stanford.edu/~vakil/216blog

[2] Qing Liu (2002) *Algebraic Geometry and Arithmetic Curves*, Oxford Graduate Texts in Mathematics, Oxford University Press.